

LOCAL BRST COHOMOLOGY IN (NON-)LAGRANGIAN FIELD THEORY

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Dedicated to the 70th birthday of Igor Victorovich Tyutin

ABSTRACT. Some general theorems are established on the local BRST cohomology for not necessarily Lagrangian gauge theories. Particular attention is given to the BRST groups with direct physical interpretation. Among other things, the groups of rigid symmetries and conservation laws are shown to be still connected, though less tightly than in the Lagrangian theory. The connection is provided by the elements of another local BRST cohomology group whose elements are identified with Lagrange structures. This extends the cohomological formulation of the Noether theorem beyond the scope of Lagrangian dynamics. We show that each integrable Lagrange structure gives rise to a Lie bracket in the space of conservation laws, which generalizes the Dickey bracket of conserved currents known in Lagrangian field theory. We study the issues of existence and uniqueness of the local BRST complex associated with a given set of field equations endowed with a compatible Lagrange structure. Contrary to the usual BV formalism, such a complex does not always exist for non-Lagrangian dynamics, and when exists it is by no means unique. The ambiguity and obstructions are controlled by certain cohomology classes, which are all explicitly identified.

1. INTRODUCTION

The BRST approach provides the most systematic method, sometimes having no alternatives, for quantizing gauge systems. The BRST theory also gives a deep insight into the classical dynamics. In particular, the classical BRST complex allows one to identify the cohomological obstructions to switching on consistent interactions in field theories and provides a cohomological understanding for Noether's correspondence between symmetries and conservation laws.

The basic ingredient of the BRST theory is a homological vector field Q , called the classical BRST differential, which is to be associated, in one way or another, to the original classical system. The practices of casting classical dynamics into the BRST framework vary from *ad hoc* constructions applicable to the models of special types to uniform schemes that work equally well for any Lagrangian or Hamiltonian system. Lagrangian dynamics can always be put into the BRST framework by means of the BV formalism [1]; for any (constrained) Hamiltonian system, the BRST differential can be constructed by the BFV method [2]. Though both the

methods have enjoyed enormous attention in various reviews¹, we would like to make here some general remarks concerning these schemes of the BRST embedding. The remarks address the key inputs and outputs of the BV and BFV formalisms that are subject to a revision when the BRST formalism is extended beyond the scope of Lagrangian/Hamiltonian dynamics.

The BV scheme of the BRST embedding starts with an action functional, which is supposed to exist for the original classical equations of motion. The classical BRST differential Q , being constructed by the BV method, has a special feature: it is a Hamiltonian vector field with respect to the BV antibracket (canonical odd Poisson bracket), i.e., $Q = (S, \cdot)$. The BV master action S is constructed by solving the BV master equation $(S, S) = 0$ in the field-antifield space. The latter space is constructed by adding ghosts and antifields to the original field content. The original action functional provides the boundary condition for the master action in the extended space. Notice that the BV prescription for the field-antifield space construction essentially relies on the fact that the original equations are Lagrangian, in which case the gauge symmetry generators coincide with the generators of Noether identities (Noether's second theorem). Beyond the scope of Lagrangian dynamics, the latter property is not true anymore and one can find many counterexamples. As a result, not just the boundary condition for the master equation is problematic to define, the BV construction of the field-antifield space does not apply to a system defined by non-Lagrangian equations. So, the canonical antibracket cannot be a general tool for constructing the BRST differential beyond the class of Lagrangian systems.

Somewhat similar scenario of the BRST embedding is played in the BFV formalism. The basic input there is the (first class constrained) Hamiltonian form of classical dynamics. The BRST differential has the Hamiltonian form $Q = \{\Omega, \cdot\}$ on the ghost extension of the original phase space. The BRST charge Ω is sought from the BFV master equation $\{\Omega, \Omega\} = 0$. The boundary condition for Ω is provided by the first class constraints of the original system. The ghost extension of the original phase space appeals to the two facts: (i) the original phase space has been already equipped with a Poisson bracket such that the evolutionary equations are Hamiltonian, and (ii) there is a pairing between the first class constraints and gauge symmetry generators. For a general system of (constrained) evolutionary equations either assumption may be invalid, so that the BFV framework cannot be directly applied to general dynamical systems.

It should be noted that in classical theory, the Hamiltonian structure of the BRST differential Q is not always relevant. Many of the deliverables of classical BRST complex appeal to its

¹For a systematic introduction to the field, we refer to the book [3].

properness (see Definition 2.2 in the next section) rather than to the Hamiltonian form of the BRST differential. The BFV/BV *quantization*, however, essentially employs the (anti)bracket, which is ‘a must’ ingredient of the quantum BRST operator.

In the papers [4], [5], [6], [7], a general scheme has been worked out for the BRST embedding and quantization of dynamical systems defined by general equations of motion, not necessarily Lagrangian or Hamiltonian. In [4], a ghost extension of the original configuration space of fields has been proposed in such a way as to implement the equations of motion through the cohomology of a proper BRST differential Q . The differential is iteratively constructed by solving the equation $Q^2 = 0$ with the usual tools of homological perturbation theory [3], [8], [9]. In the irreducible case, the boundary condition for Q is provided by the classical equations of motion together with the generators of their gauge symmetries and Noether identities, with no pairing assumed between the two groups of generators. This construction needs no bracket to define the BRST differential, and it is sufficient to develop many of the standard applications of the BRST theory to classical dynamics.

In [4], [5], two extra structures were proposed to define a consistent quantization of non-Hamiltonian/non-Lagrangian dynamics. These are called respectively the weak Poisson structure [4] and the Lagrange structure [5]. From the pure algebraic viewpoint, either structure extends the classical BRST differential Q , respectively, to P_∞ - or S_∞ -algebra², where Q is taken to be the first structure map. The second structure map defines a bracket (even for P_∞ and odd for S_∞) which is compatible with the BRST differential and satisfies the Jacobi identity up to a homotopy correction governed by Q . The usual BV and BFV formalisms fit in this algebraic picture as very special cases where the structure maps P_k and S_k vanish for all $k > 2$ and the brackets associated to P_2 and S_2 are non-degenerate.

It should be noted that the existence of a compatible Lagrange structure is a much more relaxed condition for the classical equations of motion than the requirement to be derivable from the action principle. Many examples are known of non-Lagrangian field theories admitting nontrivial Lagrange structures that allow one to construct a reasonable quantum theory [5], [6], [7], [11], [12], [13], [14]. In all the examples, the Lagrange structure enjoys the property of space-time locality.

The concept of space-time locality is of a paramount importance for quantum field theory. For a long time, the locality of the BRST differential remained a plausible hypothesis. It has been proven later [15] that this hypothesis is actually superfluous: no obstructions can appear to the existence of a local BV master action or a BRST charge provided the original classical

²Pure algebraic definitions of P_∞ and S_∞ can be found in [10].

dynamics are local and regular. Further studies of the local BRST cohomology revealed its remarkable properties together with many important applications to field-theoretical problems, see [16] for review. In particular, imposing the locality condition gives birth to many interesting groups of the BRST cohomology (e.g. those describing the rigid symmetries and conservation laws), which otherwise vanish identically.

As the BRST theory is now available for not necessarily Lagrangian dynamics, it becomes an issue to study the local BRST cohomology for this more general class of systems. It is the topic we consider in this paper. The addressed issues include all the main problems concerning the local BRST cohomology solved earlier in the Lagrangian setting, some of which are posed in a slightly different way, and some others result in different conclusions. The main results can be summarized as follows:

- A local classical BRST differential can always be defined for any local gauge theory, be it Lagrangian or not. Incorporation of a nontrivial Lagrange structure in the BRST complex is generally obstructed by some classes of the local BRST cohomology. We identify these classes and show that their number is finite. Thus, unlike the usual BRST theory for Lagrangian gauge systems, the space-time locality of non-Lagrangian equations of motion and a compatible Lagrange structure is not generally sufficient for the existence of a local BRST complex.
- Certain vanishing theorems for and general relations between various local BRST cohomology groups are established for general non-Lagrangian gauge theories. Some of these groups admit immediate physical interpretations linking them to the spaces of characteristics (conservation laws), rigid symmetries, and Lagrange structures.
- A cohomological version of Noether's first theorem, connecting conservation laws to rigid symmetries, is formulated and proven for not necessarily Lagrangian gauge theories endowed with a Lagrange structure. For general non-Lagrangian theories, the rigid symmetries and conserved currents are not so tightly related to each other as in the Lagrangian case, and a particular form of this relation strongly depends on the choice of Lagrange structure.
- We extend the Dickey bracket of conserved currents to the case of non-Lagrangian gauge theories equipped with integrable Lagrange structures. Furthermore, the Lagrange structure is shown to provide a homomorphism of the Dickey algebra of conservation laws to the Lie algebra of rigid symmetries. We also introduce the higher derived brackets that allow one to generate new characteristics, symmetries, and Lagrange structures from previously known characteristics.

The paper is organized as follows. In the next section, we briefly review the definition of the BRST complex that serves any gauge theory, be it Lagrangian or not. A special emphasis is placed on space-time locality, including the definitions of local p -forms and functionals of fields the BRST differential acts upon. Section 3 provides some auxiliary facts from homological algebra we use to study various local BRST cohomology groups. In Section 4, we prove several general theorems on the local BRST cohomology. The use of spectral sequence arguments considerably shortens the proofs as compared to the previously known Lagrangian counterparts. In Section 5, we elaborate on physical interpretation of certain groups of local BRST cohomology. These are the groups that are naturally identified with the spaces of rigid symmetries, conservation laws, and Lagrange structures. Here we also elucidate the structure of the BRST differential, more precisely its Koszul-Tate part, from the viewpoint of the underlying gauge dynamics. In Section 6, we discuss some natural operations one can define for the local BRST cohomology, the most important of which is the Dickey bracket on the space of conservation laws. Section 7 deals with the issues of existence and uniqueness for the local BRST complex. Contrary to the usual BV formalism, the existence of local BRST complex is generally obstructed by certain classes of the local BRST cohomology, which we explicitly identify. In the last Section 8, we summarize our results. Appendix A contains some technical details taken out of Sections 2 and 6.

2. A NON-LAGRANGIAN BRST COMPLEX

This section provides a glossary on the BRST theory of non-Lagrangian gauge systems with special emphasis on space-time locality. For a systematic account of the theory as well as its application to the path-integral quantization of various non-Lagrangian gauge models the reader is referred to the original papers [5], [6], [7], [11], [14].

In the local setting, a *proper gauge system of type* (n, m) is defined by the following data.

Source: A smooth manifold X of dimension Δ endowed with a suitable volume form v . We will write $\{x^\mu\}_{\mu=1}^\Delta$ for a system of local coordinates on X and denote by $\Lambda = \bigoplus_p \Lambda^p(X)$ the exterior algebra of differential forms on X .

Target: The cotangent bundle T^*M of a graded supermanifold endowed with the canonical symplectic structure. The local coordinates on the base M and the linear coordinates on the fibers are denoted respectively by

$$\varphi^I = (\varphi^{i_k}, \varphi_{i_l}), \quad \bar{\varphi}_I = (\bar{\varphi}_{i_k}, \bar{\varphi}^{i_l}), \quad k = 0, \dots, m, \quad l = 1, \dots, n+1, \quad (1)$$

so that the canonical Poisson bracket on T^*M is given by

$$\{\varphi^I, \bar{\varphi}_J\} = \delta_J^I. \quad (2)$$

We will also use the collective notation $\phi^A = (\varphi^I, \bar{\varphi}_J)$ for the whole set of local coordinates on T^*M . The various gradings prescribed to the coordinates (1) are collected in Table 1.

ghost number	momentum degree	resolution degree	pure ghost number
gh $\varphi^{i_k} = k$	Deg $\varphi^{i_k} = 0$	deg $\varphi^{i_k} = 0$	pgh $\varphi^{i_k} = k$
gh $\bar{\varphi}_{i_k} = -k$	Deg $\bar{\varphi}_{i_k} = 1$	deg $\bar{\varphi}_{i_k} = k + 1$	pgh $\bar{\varphi}_{i_k} = 0$
gh $\varphi_{i_l} = -l$	Deg $\varphi_{i_l} = 0$	deg $\varphi_{i_l} = l$	pgh $\varphi_{i_l} = 0$
gh $\bar{\varphi}^{i_l} = l$	Deg $\bar{\varphi}^{i_l} = 1$	deg $\bar{\varphi}^{i_l} = 0$	pgh $\bar{\varphi}^{i_l} = l - 1$

TABLE 1. The gradings of local coordinates on the target space.

To avoid cumbersome sign factors we assume that the Grassmann parity of the coordinates is compatible with the ghost number, i.e., the even coordinates have even ghost numbers and the odd coordinates have odd ghost numbers. In physical terms this means that we consider gauge theories without fermionic degrees of freedom. The results are easily extended to the general case of a gauge theory with both bosonic and fermionic fields.

The four gradings of local coordinates are not actually independent. Comparing the columns of Table 1, one can see that

$$\text{pgh} + \text{Deg} = \text{gh} + \text{deg}. \quad (3)$$

Fields: Smooth maps from X to T^*M . The fields can be considered as points of an (infinite dimensional) manifold \mathcal{M} . The local coordinate systems on \mathcal{M} are identified with the local coordinate expressions of maps $\phi : X \rightarrow T^*M$. The manifold \mathcal{M} inherits the gradings and the symplectic structure of T^*M . Namely, if $\phi : X \rightarrow T^*M$ is given locally by $\phi^A(x) = (\varphi^J(x), \bar{\varphi}_J(x))$, then the symplectic 2-form on \mathcal{M} reads

$$\omega = \int_X v(\delta \bar{\varphi}_J \wedge \delta \varphi^J). \quad (4)$$

Here the wedge denotes the exterior product of variational differentials.

From the field-theoretical viewpoint, the “canonical momenta” $\bar{\varphi}_I(x)$ play the role of sources to the “genuine fields” $\varphi^I(x)$ on the space-time manifold X . If X is a manifold with boundary, some boundary conditions on ϕ ’s should be imposed. For our purposes it is sufficient to suppose

all the sources vanishing on the boundary together with all their derivatives,

$$\partial_{\mu_1} \cdots \partial_{\mu_n} \bar{\varphi}_I|_{\partial X} = 0, \quad n = 0, 1, 2, \dots \quad (5)$$

In this paper, we are interested in gauge fields whose dynamics is governed by local equations of motion. The standard way to incorporate locality is to use the formalism of jet spaces [17]. Let us regard \mathcal{M} as the space of sections of the trivial fiber bundle $E = X \times T^*M$ over X and let $J^\infty E$ denote the infinite jet bundle of E over X . Each smooth section of E , that is, a field $\phi \in \mathcal{M}$, induces the section $j^\infty \phi$ of $J^\infty E$. By a local function on $J^\infty E$ we mean the pullback of a smooth function on some finite jet bundle $J^p E$ with respect to the canonical projection $J^\infty E \rightarrow J^p E$. Similarly, by a *local function of fields* ϕ on X we will mean the pullback of a local function on $J^\infty E$ via the section $j^\infty \phi : X \rightarrow J^\infty E$. The notion of local function on X can be further extended to the notion of a *local k -form* on X by considering the differential forms on X with coefficients that are local functions of fields ϕ . In terms of local coordinates a local k -form on X reads

$$\omega = \omega(x, \phi(x), \partial_\mu \phi(x), \dots, \partial_{\mu_1} \cdots \partial_{\mu_p} \phi(x))_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}. \quad (6)$$

The local forms constitute the exterior differential algebra $\mathcal{A} = \bigoplus \mathcal{A}_{m,n}^{g,k}$ graded by the ghost number g , momentum degree m , pure ghost number n , and form degree k . The volume form v on X defines the natural isomorphism $f_v : \mathcal{A}_{m,n}^{g,0} \rightarrow \mathcal{A}_{m,n}^{g,\Delta}$. Finally, we define the *local functionals* to be the integrals over X of local Δ -forms; in doing so, two local functionals are considered to be equivalent if their integrands differ by the exterior differential of a local $(\Delta - 1)$ -form. For the functionals of positive momentum degree this equivalence relation, having the form of equality, is implicit in the boundary conditions (5). Denoting the graded vector space of local functionals by $\mathcal{F} = \bigoplus \mathcal{F}_{m,n}^g$, one can think of the integration over X as a linear map

$$\int_X : \mathcal{A}_{m,n}^{g,\Delta} / d\mathcal{A}_{m,n}^{g,\Delta-1} \rightarrow \mathcal{F}_{m,n}^g.$$

This map is actually an isomorphism of vector spaces, see Theorem 4.1 below.

The Poisson bracket associated to the symplectic structure (4) equips \mathcal{F} with the structure of a graded Lie algebra. Furthermore, the bracket defines the Hamiltonian action of \mathcal{F} on the space of local functions. This action then naturally extends to the action on the whole algebra of local forms, so that we can think of \mathcal{A} as a graded module over the Lie algebra \mathcal{F} .

BRST charge: An odd local functional Ω of ghost number 1 satisfying the following two conditions:

$$\text{Deg } \Omega > 0, \quad \{\Omega, \Omega\} = 0, \quad (7)$$

plus a *properness condition* to be specified below (see Definition 2.2). The first condition in (7) can also be written as $\Omega|_{\bar{\varphi}_I=0} = 0$ and the second one is known as the *master equation*.

Define the *BRST differential* s on local functions from \mathcal{A} by

$$sA = \{\Omega, A\}, \quad \text{gh } s = 1. \quad (8)$$

By the Jacobi identity for the Poisson brackets, the BRST differential squares to zero, $s^2 = 0$, and endows the algebra \mathcal{A} with the structure of increasing cochain complex with respect to the ghost number. This complex is called the *BRST complex* of a gauge system. The action of s extends trivially from local functions to the whole algebra of local forms by setting $s(dx^\mu) = 0$ and, by the universal coefficient theorem [18], we have the isomorphism of cohomology groups

$$H^g(s, \mathcal{A}^{\bullet, k}) \cong H^g(s, \mathcal{A}^{\bullet, 0}) \otimes \Lambda^k(X).$$

Proposition 2.1. *The expansion of the BRST differential according to the resolution degree is given by*

$$s = \delta + \overset{(0)}{s} + \overset{(1)}{s} + \dots, \quad (9)$$

where

$$\deg \delta = -1, \quad \deg \overset{(n)}{s} = n, \quad \delta^2 = 0.$$

The only nontrivial part of the assertion is that the expansion (9) is bounded from below by resolution degree -1. The proof is given in Appendix A. The operator δ , called the *Koszul-Tate differential*, makes the algebra \mathcal{A} into a decreasing cochain complex with respect to the resolution degree. Denote by $H^{(r)}(\delta)$, $r \geq 0$, the corresponding cohomology groups.

Definition 2.2. *The BRST charge Ω is said to be proper if*

$$H^{(0)}(\delta) \neq 0 \quad \text{and} \quad H^{(r)}(\delta) = 0 \quad \forall r > 0.$$

In all the following, only the proper BRST charges are considered.

Consider now the expansion of the BRST charge in powers of sources. By (7) we have

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \dots, \quad \text{Deg } \Omega_n = n, \quad (10)$$

and

$$\{\Omega_1, \Omega_1\} = 0, \quad \{\Omega_1, \Omega_2\} = 0, \quad \{\Omega_2, \Omega_2\} = -2\{\Omega_1, \Omega_3\}, \quad \dots \quad (11)$$

The leading term Ω_1 is called the *classical BRST charge*. It generates the Hamiltonian vector field $s_0 = \{\Omega_1, \cdot\}$ called the *classical BRST differential*³. By virtue of the first relation in

³Notice that the differential s_0 is completely determined by its restriction onto the subalgebra of local functions with zero momentum degree. This restriction is also called the classical BRST differential [5].

(11), $s_0^2 = 0$. As for the BRST differential s , the action of s_0 extends trivially from the local functions to the local forms, giving the space \mathcal{A} the structure of cochain complex with respect to the ghost number. The differential s_0 carries all the information about the classical dynamics, hence the name. The higher order terms in the expansion (10) can be viewed as a deformation of the classical BRST charge. The deformation becomes crucial at the level of quantization. In this paper, however, our main concern will be in classical aspects of the BRST theory under consideration.

3. SOME AUXILIARY FACTS AND CONSTRUCTIONS FROM HOMOLOGICAL ALGEBRA

3.1. An exact sequence for relative cohomology groups. Given a bicomplex $C = \bigoplus_{q,r} C^{q,r}$ with differentials

$$d : C^{q,r} \rightarrow C^{q+1,r} \quad \text{and} \quad \delta : C^{q,r} \rightarrow C^{q,r-1}, \quad (12)$$

one can define various cohomology groups $H(d)$, $H(\delta)$, $H(d|\delta)$, and $H(\delta|d)$. The definition of the first two groups is standard [18], while the last two groups describe the relative cohomology of d modulo δ and δ modulo d , respectively. More precisely, $H(d|\delta)$ is the ordinary cohomology group of the quotient complex $C/\delta C$ with the differential induced by d and $H(\delta|d)$ describes the cohomology of δ in the quotient C/dC .

Theorem 3.1 ([19]). *If $H^{q,r}(d) = 0$ for some value of (q, r) , there exists an exact sequence*

$$0 \rightarrow H^{q,r+1}(d|\delta) \rightarrow H^{q,r+1}(\delta|d) \rightarrow H^{q+1,r+1}(\delta) \rightarrow H^{q+1,r+1}(\delta|d) \rightarrow H^{q,r}(d|\delta) \rightarrow 0. \quad (13)$$

Corollary 3.2. *If in the exact sequence above the middle group $H^{q+1,r+1}(\delta)$ is trivial, then*

$$H^{q,r+1}(d|\delta) \cong H^{q,r+1}(\delta|d) \quad \text{and} \quad H^{q+1,r+1}(\delta|d) \cong H^{q,r}(d|\delta).$$

Corollary 3.3. *If $H^{q,r}(d) = H^{q+1,r+1}(\delta) = H^{q+1,r}(d) = H^{q+2,r+1}(\delta) = 0$, then*

$$H^{q+1,r+1}(d|\delta) \cong H^{q,r}(d|\delta).$$

Corollary 3.4. *If $H^{q,r}(d) = H^{q+1,r+1}(\delta) = H^{q,r-1}(d) = H^{q+1,r}(\delta) = 0$, then*

$$H^{q+1,r+1}(\delta|d) \cong H^{q,r}(\delta|d).$$

3.2. Spectral sequence of filtered complex. A decreasing filtration of a complex (C, d) of vector spaces is given by a family of subspaces $F^p C^k$ (one for each k) satisfying the following conditions:

$$\begin{aligned} \dots \supset F^{p-1} C^k \supset F^p C^k \supset F^{p+1} C^k \supset \dots, \\ \bigcap_p F^p C^k = 0, \quad C^k = \bigcup_p F^p C^k, \quad d(F^p C^k) \subset F^p C^{k+1}. \end{aligned}$$

To each filtered complex one can associate a sequence of bigraded differential vector spaces $\{E_r, d_r\}_{r=0}^\infty$, called the spectral sequence, with the property that $E_0^{p,q} = F^p C^{p+q} / F^{p+1} C^{p+q}$ and $E_{r+1} = H(d_r)$. The differentials $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ are induced in a certain natural way by d . In particular, d_0 is given by the standard differential in the quotient complex $F^p C / F^{p+1} C$. The spectral sequence machinery is designed to compute the cohomology of filtered complexes. Namely, under certain regularity conditions, one can prove that $H^k(d) \cong \bigoplus_{p+q=k} E_\infty^{p,q}$. (Intuitively, the terms of the spectral sequence $\{E_r\}$ can be regarded as successive approximations *from above* to E_∞ .) Every so often, the sequence of bigraded vector spaces $\{E_r\}$ stabilizes for small values of r , i.e., $E_r \cong E_{r+1} \cong \dots \cong E_\infty$, in which case one says that the spectral sequence “collapses” at the r th step. For a systematic exposition of spectral sequences we refer the reader to [18].

3.3. L_∞ -algebra. An L_∞ -algebra⁴ is a graded vector space V endowed with multibrackets $L_n \in \text{Hom}_1(S^n V, V)$, $n \in \mathbb{N}$, satisfying the generalized Jacobi identities

$$\sum_{k+l=n} \sum_{(k,l)\text{-shuffle}} (-1)^\epsilon L_{l+1}(L_k(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}) = 0, \quad \forall n \in \mathbb{N}, \quad (14)$$

where a (k, l) -shuffle is a permutation of indices $1, 2, \dots, k+l$ such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+l)$, while $(-1)^\epsilon$ is the natural sign prescribed by the sign rule for permutation of homogeneous elements $a_1, \dots, a_n \in V$.

By definition, L_0 is just a distinguished element of V . An L_∞ -algebra with $L_0 = 0$ is called *flat*. In the flat case, the generalized Jacobi identities for $n = 1, 2, 3$ can be written as

$$d^2 a = 0,$$

$$d(a, b) + (da, b) + (-1)^{\varepsilon(a)\varepsilon(b)}(db, a) = 0, \quad (15)$$

$$((a, b), c) + (-1)^{\varepsilon(b)\varepsilon(c)}((a, c), b) + (-1)^{\varepsilon(a)(\varepsilon(c)+\varepsilon(b))}((b, c), a)$$

$$+ dL_3(a, b, c) + L_3(da, b, c) + (-1)^{\varepsilon(a)\varepsilon(b)}L_3(db, a, c) + (-1)^{(\varepsilon(a)+\varepsilon(b))\varepsilon(c)}L_3(dc, a, b) = 0,$$

where we set $da = L_1(a)$ and $(a, b) = L_2(a, b)$. As is seen the unary bracket L_1 defines a coboundary operator on V , which is also a derivation of the binary bracket. The binary bracket, in its turn, satisfies the Jacobi identity with the homotopy correction governed by L_3 . The higher Jacobi identities impose a coherent set of restrictions on L_3 and higher homotopies.

⁴We follow here the sign conventions of [10]. A physicist-oriented discussion of the L_∞ -algebras can be found in [20].

Notice that L_2 induces the Lie algebra structure on L_1 -cohomology. The usual Lie algebras can be viewed as L_∞ -algebras with $L_k = 0, \forall k \neq 2$, and L_2 being given by the Lie bracket.

4. GENERAL THEOREMS ON LOCAL BRST COHOMOLOGY

In Section 2, we have introduced, besides the BRST differential s , two more differentials: the Koszul-Tate differential δ and the classical BRST differential s_0 . Both of the differentials commute with the exterior differential d giving the algebra \mathcal{A} two different bicomplex structures. As a result we have various “absolute” and relative cohomology groups

$$H_m^g(d)_n^k, \quad H_m^g(\delta)_n^k, \quad H_m^g(s_0)_n^k, \quad H_m^g(\delta|d)_n^k, \quad H_m^g(d|\delta)_n^k, \quad H_m^g(s_0|d)_n^k. \quad (16)$$

The differentials d , δ , and s_0 being homogenous, the elements of the groups above are represented by local forms with definite ghost number g , momentum degree m , and form degree k . The d - and δ -cohomology groups are also graded by the pure ghost number n . Sometimes it will be convenient to label the cohomology groups by the resolution degree in place of the ghost number. To distinguish between these two gradings we enclose the resolution degree r in round brackets. Equation (3) allows one to switch easily from one notation to another:

$$H_m^{(r)}(\dots)_n^k = H_m^{m-r+n}(\dots)_n^k, \quad H_m^g(\dots)_n^k = H_m^{(m-g+n)}(\dots)_n^k.$$

The aim of this section is to formulate some general theorems about the cohomology groups (16). Most of the theorems below are almost identical to those of the usual BV theory of Lagrangian gauge systems [16]. This is not surprising, as there is no great difference between Lagrangian and non-Lagrangian theories at the level of the Koszul-Tate differential. One may wonder why we are interested in the (relative) cohomology of the operators δ and s_0 , leaving aside the cohomology of the “parent” BRST differential s . The reason is twofold. On the one hand, certain cohomology groups of (16) have useful interpretations in classical field theory, as we will see in Sec. 5, and on the other hand the physically relevant cohomology of the operator s is defined in the space of *nonlocal* functionals and appears to be crucial only upon quantization.

In the rest of the paper we assume the source manifold X to be (diffeomorphic to) a contractible domain in \mathbb{R}^Δ . Under this assumption the following statement, called sometimes the “algebraic Poincaré lemma”, is true.

Theorem 4.1. *The cohomology of d in \mathcal{A} is given by*

$$H(d)^k \cong \begin{cases} \mathbb{R}, & \text{for } k = 0; \\ 0, & \text{for } 0 < k < \Delta; \\ \text{the space of local functionals } \mathcal{F}, & \text{for } k = \Delta. \end{cases}$$

Proof. The proof can be found in many places, see e.g. [16], [17]. The classes of $H(d)^0$ are represented by constant functions on X . \square

Theorem 4.2. *For all $r \geq 0$, $k < \Delta$, and $m + n + r + k > 0$ there are isomorphisms*

$$\begin{aligned} H_m^{(r+1)}(\delta|d)_n^k &\cong H_m^{(r+1)}(d|\delta)_n^k, & H_m^{(r+1)}(\delta|d)_n^{k+1} &\cong H_m^{(r)}(d|\delta)_n^k, \\ H_m^{(r+1)}(d|\delta)_n^k &\cong H_m^{(r)}(d|\delta)_n^{k-1}, & H_m^{(r+2)}(\delta|d)_n^{k+1} &\cong H_m^{(r+1)}(\delta|d)_n^k. \end{aligned}$$

Proof. Fixing the numbers m and n , consider the bicomplex (12) with $C^{k,m-g+n} = \mathcal{A}_{m,n}^{g,k}$. Then the isomorphisms follow immediately from Corollaries 3.2 - 3.4 (the conditions $r \geq 0$, $k < \Delta$, and $m + n + r + k > 0$ ensure acyclicity of d and δ). \square

Theorem 4.3. *There are isomorphisms*

$$H_0^{(1)}(d|\delta)_0^0 = 0, \quad H_0^{(1)}(\delta|d)_0^1 \cong H_0^{(0)}(d|\delta)_0^0 / \mathbb{R}. \quad (17)$$

Proof. If $[a] \in H_0^{(1)}(d|\delta)_0^0$, then $da = \delta b$ for some b . Hence, $d\delta a = 0$ and by Theorem 4.1, $\delta a = c \in \mathbb{R}$. If $c = 0$, then $a = \delta m$ because δ is acyclic in positive resolution degree. In that case a represents the zero class of $H_0^{(1)}(d|\delta)_0^0$. Therefore, $H_0^{(1)}(d|\delta)_0^0 \neq 0$ iff there exists a relative d -cocycle a with $\delta a = c \neq 0$. In that case, any δ -cocycle n in resolution degree zero is trivial for we can write it as $n = \delta(na/c)$. This contradicts to nontriviality of the group $H^{(0)}(\delta)$.

To prove the second isomorphism in (17) consider the exact sequence

$$0 \rightarrow H^{k-1,r}(\delta|d) \rightarrow H^{k-1,r}(d|\delta) \rightarrow H^{k-1,r-1}(d) \rightarrow H^{k-1,r-1}(d|\delta) \rightarrow H^{k,r}(\delta|d) \rightarrow 0. \quad (18)$$

It is obtained from (13) by interchanging the roles of d and δ . Setting $k = r = 1$ and $m = n = 0$, we get $H^{0,1}(d|\delta) = 0$ and $H^{0,0}(d) = \mathbb{R}$. Then (18) reduces to the short exact sequence

$$0 \rightarrow \mathbb{R} \rightarrow H^{0,0}(d|\delta) \rightarrow H^{1,1}(\delta|d) \rightarrow 0, \quad (19)$$

from which the desired isomorphism follows. \square

Theorem 4.4. *There are isomorphisms*

$$\begin{aligned}
H_m^{(r)}(\delta|d)_n^\Delta &\cong H_m^{(r-1)}(\delta|d)_n^{\Delta-1} \cong \dots \cong H_m^{(1)}(\delta|d)_n^{\Delta-r+1} \cong H_m^{(0)}(d|\delta)_n^{\Delta-r}, & r < \Delta; \\
H_m^{(\Delta)}(\delta|d)_n^\Delta &\cong H_m^{(\Delta-1)}(\delta|d)_n^{\Delta-1} \cong \dots \cong H_m^{(1)}(\delta|d)_n^1 \cong H_m^{(0)}(d|\delta)_n^0, & m+n > 0; \\
H_0^{(\Delta)}(\delta|d)_0^\Delta &\cong H_0^{(\Delta-1)}(\delta|d)_0^{\Delta-1} \cong \dots \cong H_0^{(1)}(\delta|d)_0^1 \cong H_0^{(0)}(d|\delta)_0^0/\mathbb{R}; \\
H_m^{(r)}(\delta|d)_n^\Delta &\cong H_m^{(r-1)}(\delta|d)_n^{\Delta-1} \cong \dots \cong H_m^{(r-\Delta+1)}(\delta|d)_n^1 \cong H_m^{(r-\Delta)}(\delta)_n^0 = 0, & r > \Delta.
\end{aligned}$$

Proof. All but two isomorphisms follow directly from Theorem 4.2. The rightmost isomorphism of the third line is established by Theorem 4.3. The other isomorphism, which is not covered by Theorem 4.2, is the last isomorphism $H_0^{(2)}(\delta|d)_0^1 \cong H_0^{(1)}(\delta)_0^0 = 0$ of the fourth line. By Theorem 4.2, we have $H_0^{(2)}(\delta|d)_0^1 \cong H_0^{(1)}(d|\delta)_0^0$ and the latter group vanishes due to Theorem 4.3. \square

Theorem 4.5. *If $n > 0$, then $H_m^{(r)}(d|\delta)_n^k = 0$ for all $k < \Delta$.*

Proof. Following the terminology of [19], we will refer to the fields with strictly positive pure ghost number as *foreground fields*, treating all the other fields as background ones. The local forms from \mathcal{A} are polynomial in the foreground fields and their derivatives. Assuming $n > 0$, define the filtered complex $F^{p+1}K^k \subset F^pK^k$, where the subspace F^pK^k consists of all local k -forms $a \in \mathcal{A}_{m,n}^{g,k}$ containing no more than $q = k - p$ derivatives of the foreground fields. Clearly, the exterior differential $d : K^k \rightarrow K^{k+1}$ preserves the filtration. Now consider the quotient complex $C = K/\delta K$ with differential induced by d . The vector spaces in question, $H_m^g(d|\delta)_n^k$, are just the cohomology groups of the complex C . The decreasing filtration on K induces a decreasing filtration on C . By definition, F^pC^k is the space of equivalence classes $a + \delta K$ of k -forms with $a \in F^pK^k$. Let $\{E_r, d_r\}$ be the spectral sequence associated to the filtered complex C . The space $E_0^{p,q}$ spans the equivalence classes $a + \delta K$ represented by local $(p+q)$ -forms a involving exactly q -derivatives of the foreground fields. To describe the action of d_0 it is convenient to decompose the exterior differential into the sum of two operators, $d = d' + d''$, where the operator d' affects only the foreground fields, while d'' deals with the background fields as well as explicit dependence of the space-time coordinates. Then the class $a + \delta K \in E_0$ is a d_0 -cocycles iff $d'a = \delta b$ for some $b \in K$. But the differential d' is known to be acyclic in all but top form-degrees. Furthermore, the corresponding homotopy h , connecting d' to the projector onto the subspace of Δ -forms, can be chosen to be commuting with δ . (For the explicit construction of such a homotopy operator see [19].) The last fact implies that h passes through the quotient $K/\delta K$ giving rise to a homotopy for d_0 . As a consequence, the term $E_1^{p,q}$

appears to be supported on the antidiagonal $p + q = \Delta$, so that the spectral sequence must of necessity collapse after the first step. Hence $H_m^{(r)}(d|\delta)_n^k \cong \bigoplus_{p+q=k} E_\infty^{p,q} = 0$ for all $k < \Delta$. \square

Theorem 4.6. *If $n > 0$ and $r > 0$, then $H^{(r)}(\delta|d)_n = 0$.*

Proof. Theorem 4.2 establishes the isomorphisms

$$H_m^{(r)}(\delta|d)_n^k \cong \begin{cases} H_m^{(r-1)}(d|\delta)_n^{k-1}, & \text{for } k > 0, \\ H_m^{(r)}(d|\delta)_n^k, & \text{for } k < \Delta, \end{cases}$$

whenever the resolution degree is positive. By Theorem 4.5, all these groups are trivial at positive pure ghost number. \square

Theorems 4.3 - 4.6 may be summarized by saying that each of the relative cohomology groups $H(\delta|d)$ and $H(d|\delta)$ is either zero or isomorphic to one of the following groups:

$$H_0^{(0)}(d|\delta)_0^0/\mathbb{R}, \quad H_m^{(0)}(d|\delta)_0^k, \quad H_m^{(r)}(d|\delta)_n^\Delta, \quad H_m^{(0)}(\delta|d)_n^q, \quad (20)$$

for $k = 0, \dots, \Delta - 1$, $q = 0, \dots, \Delta$, $m \geq 0$, $n \geq 0$, and $r \geq 0$.

Let us now turn to the cohomology of the classical BRST differential. The general expansion for s_0 with respect to the resolution degree reads

$$s_0 = \delta + \gamma + s_0^{(1)} + \dots, \quad \deg \delta = -1, \quad \deg \gamma = 0, \quad \deg s_0^{(r)} = r. \quad (21)$$

The operator γ is known as the *longitudinal differential* [3]. It implements the gauge invariance. Expanding the identity $s_0^2 = 0$ according to the resolution degree, we obtain the infinite sequence of relations

$$\delta^2 = 0, \quad [\delta, \gamma] = 0, \quad \gamma^2 = [\delta, s_0^{(1)}], \quad \dots$$

The first three of these relations mean that the action of γ descends to the cohomology of δ inducing there a coboundary operator. The corresponding complex is known as the longitudinal complex.

Theorem 4.7. *The cohomology of the classical BRST differential s_0 is given by*

$$H_m^g(s_0) \cong \begin{cases} 0, & \text{for } m > g; \\ H_m^g(\gamma, H^{(0)}(\delta)_{g-m}), & \text{for } m \leq g. \end{cases} \quad (22)$$

The group $H_m^g(\gamma, H^{(0)}(\delta)_{g-m})$ describes the cohomology of γ in the cohomology of δ .

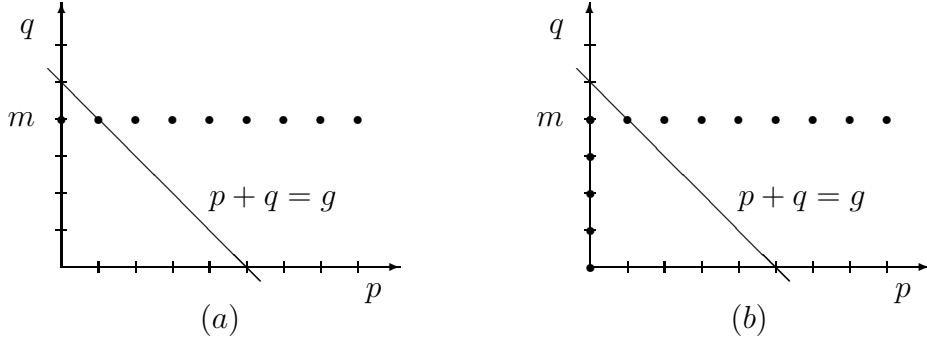


FIGURE 1. The first terms of spectral sequences from Theorems 4.7, 4.8.

Proof. In view of (3) the expansion (21) is simultaneously the expansion according to the pure ghost number,

$$\text{pgh } \delta = 0, \quad \text{pgh } \gamma = 1, \quad \text{pgh } s_0^{(r)} = r + 1.$$

Since $\text{pgh } s_0 \geq 0$ we have the filtered complex $F^{p+1}C^k \subset F^pC^k$, where $F^pC^k = \bigoplus_{n=p}^{\infty} \mathcal{A}_{m,n}^{k,\bullet}$. Define the spectral sequence $\{E_r, d_r\}$ associated to this filtration. It is clear that $E_0^{p,q} = \mathcal{A}_{m,p}^{p+q,\bullet}$ and $d_0 = \delta$. The Koszul-Tate differential being acyclic in positive resolution degree, $H_m^g(\delta)_n = 0$ for $r = n - g + m > 0$. Therefore all the non-zero terms of $E_1^{q,p} \cong H_m^{q+p}(\delta)_p$ lie on the horizontal line $q = m$, see Fig. 1a. The spectral sequence necessarily collapses at the second step; in so doing, the differential $d_1 : E_1^{p,q} \rightarrow E_1^{p,q+1}$ is induced by γ :

$$E_1^{p,q} \cong H_m^{p+q}(\delta)_p \ni [a] \mapsto d_1[a] = [\gamma a] \in H_m^{p+q+1}(\delta)_{p+1} \cong E_1^{q,p+1}.$$

Thus, $E_2^{p,q} \cong H_m^{p+q}(\gamma, H^{(0)}(\delta)_p) = 0$ unless $q = m$ and we finally get

$$H_m^g(s_0) \cong \bigoplus_{p+q=g} E_2^{p,q} \cong H_m^g(\gamma, H^{(0)}(\delta)_{g-m}).$$

Of course, $H^{(0)}(\delta)_n \equiv 0$ for $n < 0$. □

Theorem 4.8. *There are isomorphisms of the cohomology groups*

$$H_m^g(s_0|d) \cong \begin{cases} H_m^g(\delta|d)_0, & \text{for } m > g; \\ H_m^g(\gamma, H(\delta|d)_{g-m}), & \text{for } m \leq g, \end{cases} \quad (23)$$

where $H_m^p(\gamma, H(\delta|d)_{g-m})$ describes the cohomology of γ in the cohomology of δ modulo d .

Proof. We can proceed analogously to the proof of Theorem 4.7. For a fixed m , consider the filtered complex $F^{p+1}C^k \subset F^pC^k$, where $F^pC^k = \bigoplus_{n=p}^{\infty} \mathcal{A}_{m,n}^{k,\bullet}/d\mathcal{A}_{m,n}^{k,\bullet-1}$. By definition, the space $H_m^k(s_0|d)$ is the k -th cohomology group of the complex C . Associated to the filtration above is the spectral sequence $\{E_r, d_r\}$ with $E_0^{q,p} = \mathcal{A}_{m,p}^{p+q,\bullet}/d\mathcal{A}_{m,p}^{p+q,\bullet-1}$. The differential d_0 is naturally

induced by δ and we can identify $E_1^{p,q}$ with $H_m^{p+q}(\delta|d)_p$. By Theorem 4.6, $H_m^{p+q}(\delta|d)_p = 0$ whenever $p > 0$ and $q < m$. Nonnegativity of the resolution degree $r = p - (p + q) + m$ also implies that $q \leq m$. All potentially non-zero spaces of E_1 are depicted in Fig. 1b. As is seen, they are nested either on the horizontal line $q = m$ or on the vertical segment $p = 0, 0 \leq q < m$. By dimensional reasons, the spectral sequence collapses from E_2 . Notice that the differential d_1 is induced by γ and becomes zero upon restriction to the spaces $E_1^{p,q}$ with $q < m$. Therefore, $E_2^{p,q} \cong E_1^{p,q}$ for $q < m$ and we see that

$$H_m^g(s_0|d) \cong \bigoplus_{p+q=g} E_2^{p,q} \cong H_m^g(\gamma, H(\delta|d)_{g-m}) \quad \text{for } g \geq m$$

and

$$H_m^g(s_0|d) \cong \bigoplus_{p+q=g} E_2^{p,q} \cong \bigoplus_{p+q=g} E_1^{p,q} \cong H_m^g(\delta|d)_0 \quad \text{for } g < m.$$

The proof is complete. \square

The most notable among the groups (23) is $H_0^0(s_0|d)^\Delta$. It is the group that is identified with the space of physical observables of a gauge system [5].

5. INTERPRETATION OF THE GROUPS $H_{g+1}^g(\delta|d)_0^\Delta$

In this section, we consider some special groups of relative δ -cohomology with a straightforward physical interpretation. Namely, we are interested in the groups $H_{g+1}^g(\delta|d)_0^\Delta$ for small g 's. The elements of these groups are represented by local functionals in pure ghost number zero and resolution degree one. To streamline our notation, we will write $\mathcal{L}_g = \mathcal{F}_{g+1,0}^g$. Notice that the graded vector space

$$\mathcal{L} = \bigoplus_{g=-1}^{\infty} \mathcal{L}_g$$

is closed with respect to the Poisson bracket on \mathcal{F} ,

$$\{\mathcal{L}_n, \mathcal{L}_m\} \subset \mathcal{L}_{n+m}, \tag{24}$$

so that we can speak of the graded Lie algebra \mathcal{L} of depth -1 .

To clarify the structure of this algebra it is convenient to use the condensed notation introduced in [5]:

$$\begin{aligned} \phi^i &= \varphi^{i_0}(x), & \bar{\phi}_i &= \bar{\varphi}_{i_0}(x), & \eta_a &= \varphi_{i_1}(x), & \bar{\eta}^a &= \bar{\varphi}^{i_1}(x), \\ c^\alpha &= \varphi^{i_1}(x), & \bar{c}_\alpha &= \bar{\varphi}_{i_1}(x), & \xi_A &= \varphi_{i_2}(x), & \bar{\xi}^A &= \bar{\varphi}^{i_2}(x). \end{aligned}$$

The superindices on the left hand side comprise both the discrete indices of the fields and the space-time coordinates x^μ ; in so doing, a repeated superindex means usual summation over

its discrete part and integration over X with the measure v . All the partial derivatives are understood as variational ones.

With this notation the Koszul-Tate differential takes the form

$$\delta = T_a(\phi) \frac{\partial}{\partial \eta_a} + \bar{\eta}^a \partial_i T_a(\phi) \frac{\partial}{\partial \phi_i} + \eta_a Z_A^a(\phi) \frac{\partial}{\partial \xi_A} + \bar{\phi}_i R_\alpha^i(\phi) \frac{\partial}{\partial \bar{c}_\alpha} + \bar{\eta}^a \eta_b U_{\alpha a}^b(\phi) \frac{\partial}{\partial \bar{c}_\alpha} + \cdots \quad (25)$$

Here the dots stand for the terms differentiating by fields of resolution degree > 2 .

Now it is a good point to explain in which sense δ is a Koszul-Tate differential. The “Koszul part” of $\delta = \delta_K + \delta_T$ is given by the first two terms in (25) and appears to be Hamiltonian:

$$\delta_K = \{ \overset{(0)}{\Omega}, \cdot \}, \quad (26)$$

with $\overset{(0)}{\Omega} = \bar{\eta}^a T_a(\phi)$ being the resolution-degree-zero part of the BRST charge Ω .

Let us introduce the supercommutative subalgebra of local forms $\mathcal{B} \subset \mathcal{A}$ defined by the conditions

$$\deg \mathcal{B} = 0, \quad \text{pgh } \mathcal{B} = 0,$$

and let \mathcal{I} be an ideal of \mathcal{B} generated by the local functions $T_a(\phi)$ and $\bar{\eta}^a \partial_i T_a$. Then the Koszul-Tate differential δ defines a homological resolution of the quotient algebra \mathcal{B}/\mathcal{I} . Namely, let (\mathcal{A}_0, δ) be the subalgebra of the differential supercommutative algebra (\mathcal{A}, δ) constituted by the local forms of pure ghost number zero, $\text{pgh } \mathcal{A}_0 = 0$. The algebra \mathcal{A}_0 is \mathbb{N} -graded by the resolution degree and contains \mathcal{B} as the subalgebra of degree 0. Since δ is acyclic in positive resolution degree, we have

$$\begin{aligned} H_k(\mathcal{A}_0, \delta) &= 0, \quad k > 0, \\ H_0(\mathcal{A}_0, \delta) &\cong \mathcal{B}/\mathcal{I}. \end{aligned} \quad (27)$$

From the physical viewpoint, the generators of the ideal \mathcal{I} can be regarded as equations of motion for the fields ϕ^i and $\bar{\eta}^a$,

$$T_a(\phi) = 0, \quad \bar{\eta}^a \partial_i T_a(\phi) = 0. \quad (28)$$

The first set of equations describes the dynamics of the original gauge fields ϕ . The second group of equations, called adjoint, plays an auxiliary role as the fields $\bar{\eta}$'s, being of ghost number 1, are unphysical. We will refer to (28) as the *extended system* of dynamical equations. By construction, the group $H_0(\mathcal{A}_0, \delta)$ is naturally graded by the ghost number and its ghost-number-zero subgroup is isomorphic to the algebra of local forms of fields ϕ 's modulo equivalence relation: two forms are considered to be equivalent if they take the same values on each solution to the equations of motion $T_a(\phi) = 0$. Denoting by \mathcal{N} the space of fields ϕ and by $\Sigma \subset \mathcal{N}$ the subspace of solutions to the equations $T_a(\phi) = 0$, we can say that the Koszul-Tate differential

implements the restriction of the local forms of fields $\phi \in \mathcal{N}$ to the subspace $\Sigma \subset \mathcal{N}$. In the physical literature the space of solutions Σ to the classical equations of motion is called *shell*.

Notice that for the theories of type $(0, 0)$ the Koszul-Tate differential δ reduces to δ_K . In this case, the original equations of motion $T_a(\phi) = 0$ are necessarily independent and enjoy no gauge freedom. If the equations happen to be dependent (reducible), then there is an (overcomplete, in general) basis of Noether's identity generators Z_A^a such that

$$Z_A^a T_a = 0. \quad (29)$$

On the other hand, the presence of gauge symmetries for the equations $T_a = 0$ implies the existence of an (overcomplete, in general) basis of gauge symmetry generators $R_\alpha = R_\alpha^i \partial_i$ together with structure functions $U_{\alpha a}^b$ such that

$$R_\alpha^i \partial_i T_a = U_{\alpha a}^b T_b. \quad (30)$$

Unlike the Lagrangian case, where we can identify Z 's with R 's due to Noether's second theorem, there is no duality between the gauge symmetries and the Noether identities for general non-Lagrangian dynamics. The duality, however, is restored at the level of the extended system (28). Each generator of the original identities (29) is simultaneously a generator of the gauge transformation in the extended space:

$$\delta_\varepsilon \phi^i = 0, \quad \delta_\varepsilon \bar{\eta}^a = \varepsilon^A Z_A^a(\phi). \quad (31)$$

Furthermore, every gauge transformation $\delta_\varepsilon \phi^i = \varepsilon^\alpha R_\alpha^i(\phi)$ of the original equations of motion gives rise to the Noether identity for the extended system (28). (It is obtained by contracting (30) with $\bar{\eta}^a$.) This duality is an immediate consequence of the fact that the extended system of equations (28) is variational. Although the corresponding "action functional" $\bar{\Omega}^{(0)} = \bar{\eta}^a T_a$ is odd, the reasoning of the second Noether's theorem still applies to this situation. As a result, the generators of Noether identities for (28) coincide with the generators of gauge symmetries and the same is true for the structure functions defining the higher-order reducibility conditions (if any). All these structure functions, including Z 's and R 's, are incorporated in Tate's part δ_T of the Koszul-Tate differential (25).

Having explained the "physical meaning" of the Koszul-Tate differential, let us come back to the graded Lie algebra \mathcal{L} . The general element of the homogeneous subspace \mathcal{L}_g has the form

$$A = \bar{\phi}_i A_{a_1 \dots a_g}^i(\phi) \bar{\eta}^{a_1} \dots \bar{\eta}^{a_g} + \eta_a A_{a_1 \dots a_{g+1}}^a(\phi) \bar{\eta}^{a_1} \dots \bar{\eta}^{a_{g+1}}. \quad (32)$$

Observe that the restriction of the Koszul-Tate operator (25) onto the subspace \mathcal{L} is given by the Koszul differential (26). As a result, the action of $\delta|_{\mathcal{L}} = \delta_K$ is Hamiltonian and the kernel

$\mathcal{Z} = \ker \delta|_{\mathcal{L}}$ appears to be a subalgebra in the graded Lie algebra \mathcal{L} ,

$$\mathcal{Z} = \bigoplus_{g=-1}^{\infty} \mathcal{Z}_g, \quad \{\mathcal{Z}_n, \mathcal{Z}_m\} \subset \mathcal{Z}_{n+m}. \quad (33)$$

Actually, a more strong statement is true: The Lie algebra structure on \mathcal{Z} descends to the δ -cohomology. This means that all the δ -coboundaries from \mathcal{Z} form an ideal in the Lie algebra \mathcal{Z} , so that it makes sense to speak of the Lie algebra structure on the quotient space

$$\mathcal{Z}/(\mathcal{Z} \cap \delta\mathcal{F}) \cong \bigoplus_{g=-1}^{\infty} H_{g+1}^g(\delta|d)_0^{\Delta}.$$

The last fact is not obvious at all as the space $\mathcal{Z} \cap \delta\mathcal{F}$ is not exhausted by $\delta\mathcal{L}$, while the action of δ on the whole of \mathcal{F} is non-Hamiltonian. A rigorous definition of the Lie bracket on $\mathcal{Z}/(\mathcal{Z} \cap \delta\mathcal{F})$ will be given in Sec. 6.

The element (32) of \mathcal{L}_g is a δ -cocycle iff

$$\delta A = (\partial_i T_{a_1} A_{a_2 \dots a_{g+1}}^i + T_a A_{a_1 \dots a_{g+1}}^a) \bar{\eta}^{a_1} \dots \bar{\eta}^{a_{g+1}} = 0. \quad (34)$$

Since the ghost fields $\bar{\eta}$'s are all odd and independent, the last condition implies that

$$\partial_i T_{[a_1} A_{a_2 \dots a_{g+1}}^i] + T_a A_{a_1 \dots a_{g+1}}^a = 0, \quad (35)$$

where the square brackets mean antisymmetrization in the usual (i.e., non-graded) sense. The cocycle A is a δ -coboundary if there is a local functional

$$\begin{aligned} B &= \bar{\phi}_i \bar{\phi}_j B_{a_1 \dots a_{g-1}}^{ij} \bar{\eta}^{a_1} \dots \bar{\eta}^{a_{g-1}} + \bar{\phi}_i \eta_a B_{a_1 \dots a_g}^{ai} \bar{\eta}^{a_1} \dots \bar{\eta}^{a_g} + \eta_a \eta_b B_{a_1 \dots a_{g+1}}^{ab} \bar{\eta}^{a_1} \dots \bar{\eta}^{a_{g+1}} \\ &+ \bar{c}_\alpha B_{a_1 \dots a_g}^\alpha \bar{\eta}^{a_1} \dots \bar{\eta}^{a_g} + \xi_A B_{a_1 \dots a_{g+1}}^A \bar{\eta}^{a_1} \dots \bar{\eta}^{a_{g+1}} \end{aligned} \quad (36)$$

such that $A = \delta B$. Explicitly,

$$\begin{aligned} A_{a_1 \dots a_g}^i &= 2\partial_j T_{[a_1} B_{a_2 \dots a_g]}^{ij} + T_a B_{a_1 \dots a_g}^{ai} + R_\alpha^i B_{a_1 \dots a_g}^\alpha, \\ A_{a_1 \dots a_{g+1}}^a &= \partial_i T_{[a_1} B_{a_2 \dots a_{g+1}}^{ai}] - 2T_b B_{a_1 \dots a_{g+1}}^{ab} - U_{\alpha[a_1} B_{a_2 \dots a_{g+1}}^\alpha] + Z_A^a B_{a_1 \dots a_{g+1}}^A. \end{aligned} \quad (37)$$

Consider now the condition (34) for $g = -1, 0, 1, 2$.

5.1. The space of characteristics is $H_0^{-1}(\delta|d)_0^{\Delta}$. For $g = -1$ the cocycle condition (34) reduces to⁵

$$A^a T_a = \int_X d\omega \quad (38)$$

⁵So far we have considered local functionals modulo boundary terms, that is, integrals of total derivatives. The total derivatives, however, cannot be ignored when discussing the conservation laws; hence, we write them explicitly here.

for some local $(\Delta - 1)$ -form ω . According to the condensed notation, the l.h.s. of (38) is given by the integral of a linear differential operator acting on the equations of motion. Whenever the result of such an action is an exact Δ -form, one refers to $A = \{A^a\}$ as the generator of *identities* for the equations of motion $T_a(\phi) = 0$. The trivial cocycles (37) correspond to linear combinations of trivial and Noether's identities, namely,

$$A^a = 2T_b B^{ba} + Z_A^a B^A.$$

Since $d\omega$ vanishes on the shell Σ , the $(\Delta - 1)$ -form ω gives rise to the conserved current $j = *\omega$, where the Hodge operator $*$: $\Lambda^p(X) \rightarrow \Lambda^{\Delta-p}(X)$ is defined by an appropriate metric on X .

The quotient of the whole space of identities by the Noether and trivial identities is known as the space of characteristics $\text{Char}(T)$ for the equations $T_a(\phi) = 0$. This leads us to the following identification:

$$\text{Char}(T) = H_0^{-1}(\delta|d). \quad (39)$$

Notice that Rel.(38) does not specify the current $j = *\omega$ unambiguously, since one can add to ω any closed (and hence exact) $(\Delta - 2)$ -form $d\alpha$ as well as any local $(\Delta - 1)$ -form γ that is proportional to the equations of motion and their differential consequences. (The latter redefinition can be absorbed by the l.h.s. of (38).) Modding out by these ambiguities, we get a well-defined class $[\omega] \in H_0^0(d|\delta)_0^{\Delta-1}$ associated to a characteristic $[A] \in H_0^{-1}(\delta|d)_0^\Delta$. The assignment $[A] \mapsto [\omega]$ defines the bijection between the groups $H_0^{-1}(\delta|d)_0^\Delta$ and $H_0^0(d|\delta)_0^{\Delta-1}/\delta_{\Delta,1}\mathbb{R}$ stated by Theorem 4.4. In physical terms, the latter group can be identified with the space of nontrivial conservation laws $\text{CL}(T)$ for the equations of motion $T_a(\phi) = 0$. Thus, we arrive at the following isomorphism:

$$\text{Char}(T) \cong \text{CL}(T). \quad (40)$$

5.2. The space of rigid symmetries is $H_1^0(\delta|d)_0^\Delta$. For $g = 0$ the cocycle condition (35) takes the form

$$A^i \partial_i T_a + A_a^b T_b = 0. \quad (41)$$

It means that the vector field $A = A^i \partial_i$ acting on the space \mathcal{N} of fields ϕ defines a symmetry of the equation of motion. The set of all the symmetries $\text{Sym}'(T)$ form a Lie algebra with respect to the commutator of vector fields. The trivial symmetries (i.e., vector fields vanishing on the shell Σ) and the gauge symmetries correspond to the coboundaries (37):

$$A^i = B^\alpha R_\alpha^i + T_a B^{ai}, \quad A_a^b = \partial_i T_a B^{bi} - 2T_c B_a^{cb} - U_{\alpha a}^b B^\alpha + Z_A^b B_a^A.$$

The space of rigid (or global) symmetries $\text{RSym}(T)$ is defined to be the quotient of all the symmetries by trivial and gauge symmetries. This leads to the identification

$$\text{RSym}(T) = H_1^0(\delta|d)_0^\Delta. \quad (42)$$

Notice that the homogeneous subspace $\mathcal{Z}_0 \subset \mathcal{Z}$ is also a Lie algebra. If

$$A^1 = \bar{\phi}_i A_1^i + \eta_a A_{1b}^a \bar{\eta}^b, \quad A^2 = \bar{\phi}_i A_2^i + \eta_a A_{2b}^a \bar{\eta}^b$$

are two relative δ -cocycles associated to symmetries $A_1, A_2 \in \text{Sym}'(T)$, then the Poisson bracket $\{A^1, A^2\}$ is again a relative δ -cocycle associated to the commutator of the vector fields $[A_1, A_2] \in \text{Sym}'(T)$. Thus, we have an isomorphism between the Lie algebras \mathcal{Z}_0 and $\text{Sym}'(T)$. As mentioned above, this isomorphism descends to the cohomology, inducing a Lie algebra structure on the space of all rigid symmetries (42).

5.3. The space of Lagrange structures is $H_2^1(\delta|d)_0^\Delta$. The elements of $H_2^1(\delta|d)_0^\Delta$ are represented by the local functionals

$$A = \bar{\phi}_i A_a^i(\phi) \bar{\eta}^a + \eta_c A_{ab}^c(\phi) \bar{\eta}^a \bar{\eta}^b, \quad (43)$$

where the structure functions A_a^i and A_{ab}^c obey the cocycle condition

$$A_a^i \partial_i T_b - A_b^i \partial_i T_c + 2A_{ab}^c T_c = 0. \quad (44)$$

The last equation is nothing but the defining relation for a *Lagrange structure* [5]. The set of vector fields $A_a = A_a^i \partial_i$ on \mathcal{N} is called the *Lagrange anchor*. The trivial cocycles (37) correspond to the trivial Lagrange structures with

$$A_a^i = 2\partial_j T_a B^{ji} + T_b B_a^{bi} + R_\alpha^i B_a^\alpha, \quad A_{ab}^c = \partial_i T_{[a} B_{b]}^{ci} - 2T_d B_{ab}^{cd} - U_{\alpha[a}^c B_{b]}^\alpha + Z_A^c B_{ab}^A. \quad (45)$$

The group $H_2^1(\delta|d)_0^\Delta$ is thus naturally identified with the space of local Lagrange structures modulo trivial ones,

$$\text{LS}(T) = H_2^1(\delta|d)_0^\Delta. \quad (46)$$

We know that the Poisson square of the cocycle (43) representing a Lagrange structure $[A]$ is a relative δ -cocycle from \mathcal{Z}_2 . This cocycle may well be nontrivial.

Definition 5.1. A Lagrange structure $[A] \in H_2^1(\delta|d)_0^\Delta$ is said to be integrable if

$$[\{A, A\}] = 0 \in H_3^2(\delta|d)_0^\Delta.$$

The reason for introducing the notion of integrability is twofold. For one thing, each BRST charge involves an integrable Lagrange structure; for another, each integrable Lagrange structure gives rise to a Lie bracket on the space of conservation laws together with a Lie algebra homomorphism to the space of rigid symmetries. Both of these statements will be detailed in the next two sections. Definition 5.1 suggests to interpret the group $H_3^2(\delta|d)_0^\Delta$ as the *space of obstructions to integrability*.

Not so much experience has been gained yet of computing the group $\text{LS}(T)$ even for linear equations of motion. In the recent paper [21] it is shown that the study of local BRST cohomology for not necessarily Lagrangian gauge theories can be always reduced to the case with the classical BRST differential involving only the first space-time derivatives of fields. This first order reduction is achieved by introducing not necessarily finite number of auxiliary fields. It is worth to mention in this connection the work [13] devoted to the study of local BRST cohomology in the AKSZ sigma-model [22]. The point is that the AKSZ sigma-model, being dynamically empty in the bulk, defines classical dynamics and quantization of the boundary degrees of freedom⁶. In particular, the corresponding topological action should involve some Lagrange structure for the boundary dynamics. It was shown [13] that whenever the AKSZ sigma-model has a *finite dimensional* target space, the group $\text{LS}(T)$ appears to be isomorphic to the space of bi-vectors on the target space. The integrability condition (5.1) amounts then to the Jacobi identity for the corresponding bi-vector. In principle, any field theory can be brought to the FDA form at the cost of introducing an *infinite dimensional* target space. In that case, however, the group $\text{LS}(T)$ is far from being exhausted by the bi-vectors. What is more, the physically relevant Lagrange structures do not reduce to the target space bi-vectors even for the FDA form of the d'Alembert equation [14].

6. MULTIPLICATIVE STRUCTURES

Theorem 4.8 establishes an isomorphism

$$\pi : H_m^g(s_0|d) \rightarrow H_m^g(\delta|d)_0, \quad m > g, \quad (47)$$

between the relative cohomology groups. A particular realization of this isomorphism is as follows. Take a relative s_0 -cocycle a representing a class of $H_m^g(s_0|d)$ and expand it according

⁶It is appropriate to note that every field theory in d dimensions, be it Lagrangian or not, can be converted into an equivalent topological Lagrangian model in $d+1$ dimensions following the systematic procedure proposed in Ref. [5]. Whenever the field equations in d dimensional space (considered as the boundary of $d+1$ dimensional bulk) have the form of free differential algebra (FDA), the corresponding $d+1$ topological Lagrangian theory has the form of the AKSZ sigma-model.

to the pure ghost number,

$$a = a_0 + a_1 + \cdots, \quad \text{pgh } a_n = n.$$

Then a_0 is a relative δ -cocycle representing a class of $H_m^g(\delta|d)_0$. Conversely, any cohomology class $[a_0] \in H_{g+1}^g(\delta|d)_0$ is uniquely extended to the class $[a] \in H_{g+1}^g(s_0|d)$ by adding terms of positive pure ghost number to the representative cocycle a_0 .

Below we apply the isomorphism (47) to define various multiplicative structures on the group

$$H^{(1)}(\delta|d)_0^\Delta = \bigoplus_{g=-1}^{\infty} H_{g+1}^g(\delta|d)_0^\Delta,$$

some of which have been already appeared in the previous section.

With the Hamiltonian action of the classical BRST differential $s_0 = \{\Omega_1, \cdot\}$, the groups $H_{g+1}^g(s_0|d)^\Delta$ form a graded Lie algebra with respect to the Poisson bracket: for any $[a] \in H_{g+1}^g(\delta|d)_0^\Delta$ and $[b] \in H_{g'+1}^{g'}(\delta|d)_0^\Delta$ we have

$$\{[a], [b]\} = [\{a, b\}] \in H_{g+g'+1}^{g+g'}(s_0|d)^\Delta.$$

The isomorphism (47) allows one to transfer this Lie algebra structure to the group $H^{(1)}(\delta|d)_0^\Delta$ by setting

$$\{\alpha, \beta\} = \pi(\{\pi^{-1}(\alpha), \pi^{-1}(\beta)\}), \quad \forall \alpha, \beta \in H^{(1)}(\delta|d)_0^\Delta. \quad (48)$$

The validity of the Jacobi identity is obvious. Here we deliberately denote the pulled back Lie bracket on $H^{(1)}(\delta|d)_0^\Delta$ by braces. The reason is that the right hand side of (48) coincides exactly with the cohomology class of the Poisson bracket of relative δ -cocycles representing the classes α and β . The proof of this fact is given in Appendix A. Thus, one can multiply (representatives of) the classes α and β as such, i.e., omitting the maps π^{-1} and π . Formula (48) just explains why the result of multiplication does not depend on the choice of representative cocycles.

The Lie bracket on $H^{(1)}(\delta|d)_0^\Delta$ gives rise to a rich variety of interesting multiplicative structures on the spaces of characteristics (39), rigid symmetries (42) and Lagrange structures (46). First of all, the rigid symmetries, being supported in ghost number zero, form a closed Lie algebra with respect to the Poisson bracket. They also act on the spaces of characteristics and Lagrange structures in the Hamiltonian way:

$$\text{RSym}(T) \times \text{Char}(T) \rightarrow \text{Char}(T) : \quad (X, \Psi) \mapsto \{X, \Psi\},$$

$$\text{RSym}(T) \times \text{LS}(T) \rightarrow \text{LS}(T) : \quad (X, \Lambda) \mapsto \{X, \Lambda\}.$$

So, we can regard the spaces $\text{Char}(T)$ and $\text{LS}(T)$ as modules over the Lie algebra $\text{RSym}(T)$. Notice that unlike the Lagrange structures, the characteristics form an abelian Lie algebra with respect to the Poisson bracket.

Consider now the bilinear map

$$\text{LS}(T) \times \text{Char}(T) \rightarrow \text{RSym}(T) : \quad (\Lambda, \Psi) \mapsto \{\Lambda, \Psi\},$$

which assigns to a Lagrange structure Λ and a characteristic Ψ the rigid symmetry $X_\Psi = \{\Lambda, \Psi\}$. The rigid symmetries belonging to the image of this map are called *characteristic symmetries*. Fixing the Lagrange structure Λ , we obtain a linear map $f_\Lambda = \{\Lambda, \cdot\}$ from the space of characteristics to the space of rigid symmetries. The homomorphism

$$f_\Lambda : \text{Char}(T) \rightarrow \text{RSym}(T) \tag{49}$$

can be regarded as a systematic extension of the first Noether theorem to non-Lagrangian equations of motion ⁷. Indeed, on account of (40), we deduce that any conservation law gives rise to a characteristic symmetry. Moreover, for the Lagrangian systems endowed with the *canonical* Lagrange structure, the map (49) is injective and its image coincides with the rigid symmetries that preserve the corresponding action functional [12]. In the general case, the homomorphism (49) is neither injective nor surjective.

The natural question arises whether it is possible to pull back the Lie algebra structure to the space of characteristics via the homomorphism (49). It turns out that such a Lie algebra structure on $\text{Char}(T)$ does exist whenever Λ is integrable in the sense of Definition 5.1. The corresponding Lie bracket on characteristics can be defined as the derived bracket

$$(\Psi_1, \Psi_2)_\Lambda = \{\{\Lambda, \Psi_1\}, \Psi_2\}, \quad \forall \Psi_1, \Psi_2 \in \text{Char}(T). \tag{50}$$

Notice that

$$\text{gh}(\cdot, \cdot)_\Lambda = 1, \quad \text{Deg}(\cdot, \cdot)_\Lambda = 0.$$

The Jacobi identity for this bracket follows from two facts: (i) the characteristics form an abelian algebra with respect to the Poisson bracket and (ii) the Poisson square of the Lagrange structure is equal to zero, $\{\Lambda, \Lambda\} = 0$. From the Jacobi identity for the Poisson bracket it also follows that

$$(\Psi_1, \Psi_2)_\Lambda = -(\Psi_2, \Psi_1)_\Lambda.$$

⁷For a modern exposition of the Noether theorem, including historical background and miscellaneous generalizations, we refer the reader to the recent book [23].

Now one can verify that (49) is a Lie algebra homomorphism. Indeed, if $X_{\Psi_1} = \{\Lambda, \Psi_1\}$ and $X_{\Psi_2} = \{\Lambda, \Psi_2\}$, then

$$\begin{aligned} \{X_{\Psi_1}, X_{\Psi_2}\} &= \{\{\Lambda, \Psi_1\}, \{\Lambda, \Psi_2\}\} = \{\Lambda, \{\{\Lambda, \Psi_1\}, \Psi_2\}\} - \{\{\Lambda, \{\Lambda, \Psi_1\}\}, \Psi_2\} \\ &= \{\Lambda, (\Psi_1, \Psi_2)_\Lambda\} - \frac{1}{2}\{\{\{\Lambda, \Lambda\}, \Psi_1\}, \Psi_2\} = X_{(\Psi_1, \Psi_2)_\Lambda}. \end{aligned}$$

The Lie algebra structure on the space of conservation laws was first introduced by Dickey in the context of Lagrangian field theory without gauge symmetry [24]. The extension to Lagrangian gauge theories has been found by Barnich and Henneaux [25]. On account of the isomorphism (40) one can view the Lie bracket (50) as a further generalization of the Dickey algebra to the case of not necessarily Lagrangian gauge theories endowed with an integrable Lagrange structure.

It should be noted that the homomorphism

$$(\cdot, \cdot)_\Lambda : \text{Char}(T) \times \text{Char}(T) \rightarrow \text{Char}(T)$$

maps characteristics to characteristic for any Lagrangian structure Λ , be it integrable or not. Integrability is only crucial for the bracket (50) to meet the Jacobi identity. Leaving aside the Jacobi identity, one can regard (50) as a special case of a more general construction that associates a family of multi-brackets to any element $\Theta \in H_m^{m-1}(\delta|d)_0^\Delta$. These multi-brackets are given by

$$(\Psi_1, \dots, \Psi_{m-k})_\Theta^k = \{\dots \{\{\Theta, \Psi_1\}, \Psi_2\}, \dots, \Psi_{m-k}\}, \quad 0 \leq k < m.$$

Setting $k = 0, 1, 2$ we get the multi-linear maps

$$\begin{aligned} (\dots)_\Theta^0 &: \text{Char}(T)^{\times m} \rightarrow \text{Char}(T), \\ (\dots)_\Theta^1 &: \text{Char}(T)^{\times m-1} \rightarrow \text{RSym}(T), \\ (\dots)_\Theta^2 &: \text{Char}(T)^{\times m-2} \rightarrow \text{LS}(T). \end{aligned}$$

The last formulae provide a useful interpretation of the higher cohomology groups $H_m^{m-1}(\delta|d)_0^\Delta$ as multi-linear operations on characteristics.

A somewhat different family of multi-brackets, generalizing the bracket (50), is generated by the BRST charge itself. Consider the expansion (10) of the BRST charge according to the momentum degree. The second equation in (11) says that the term Ω_2 is a s_0 -cocycle. Hence, it defines a class $[\Omega_2] \in H_2^1(s_0|d)^\Delta$ and, via the isomorphism (47), a Lagrange structure $\pi([\Omega_2]) \in H_2^1(\delta|d)_0^\Delta$. Due to the third relation in (11), the Lagrange structure $\Lambda = \pi([\Omega_2])$ is integrable and defines the Lie bracket (50) on the space of characteristics. One can put this Lie algebra structure on $\text{Char}(T)$ in a more wide context of L_∞ -algebras (see Section 3.3). In

[5], it was noticed that any BRST charge associated to a gauge system gives rise to an L_∞ -structure on the space $\mathcal{F}_{0,\bullet}$ of local functionals in momentum degree zero. The corresponding multibrackets $L_n : \mathcal{F}_{0,\bullet} \rightarrow \mathcal{F}_{0,\bullet}$ are defined as the higher derived brackets [10]:

$$L_n(a_1, \dots, a_n) = \{ \dots \{ \{ \Omega_n, a_1 \}, a_2 \}, \dots, a_n \}. \quad (51)$$

Using the Jacobi identity for the Poisson bracket and commutativity of the Poisson algebra $\mathcal{F}_{0,\bullet}$, one can see that the multibrackets (51) are graded symmetric and satisfy the generalized Jacobi identities (14) by virtue of the master equation $\{\Omega, \Omega\} = 0$. It then follows from Rels. (15) that the binary bracket

$$L_2(a_1, a_2) = \{ \{ \Omega_2, a_1 \}, a_1 \}$$

induces a Lie bracket in the cohomology of the coboundary operator $L_1 = \{ \Omega_1, \cdot \} : \mathcal{F}_{0,\bullet} \rightarrow \mathcal{F}_{0,\bullet}$. This yields a Lie algebra structure on $H_0(s_0|d)^\Delta$. The space of characteristics, being isomorphic to $H_0^{-1}(s_0|d)^\Delta$, constitutes a subalgebra in the Lie algebra $H_0(s_0|d)^\Delta$. It is clear that restricting the induced Lie bracket onto $H_0^{-1}(s_0|d)^\Delta \cong H_0^{-1}(\delta|d)_0^\Delta$, we arrive at the bracket (50) with $\Lambda = \pi([\Omega_2])$. Thus, one can speak of a canonical Lie algebra isomorphism (49) associated to a given BRST charge.

7. EXISTENCE AND UNIQUENESS OF THE BRST CHARGE

So far we have considered the BRST charge as given from the outset, and the main focus has been placed on the study of the corresponding BRST cohomology. We have shown that some of the local BRST cohomology groups have a direct interpretation in terms of the underlying classical dynamics. In practice, the problem set up is quite opposite: given the classical equations of motion, it is necessary to construct a proper BRST charge. This problem is similar to that in the BV quantization of Lagrangian gauge systems where one needs to construct a local master action starting from a gauge invariant action functional. The well known theorem [15] ensures the unique existence of the local master action under some regularity conditions. These conditions are imposed to provide the existence of the Koszul-Tate resolution for the stationary surface defined by the equations of motion. The Koszul-Tate differential appears thus the only input that comes from a classical theory both in the Lagrangian and non-Lagrangian settings. The inductive construction of the Koszul-Tate resolution for the stationary surface of Lagrangian equations of motion has been proposed in [8], [9]. It can be carried over immediately to non-Lagrangian gauge theories if one starts with the extended system of dynamical equations (28) governed by the “odd action” $\overset{(0)}{\Omega} = \bar{\eta}^a T_a$. A non-Lagrangian counterpart of Henneaux’s theorem then reads:

Theorem 7.1. *Up to a canonical transformation any classical BRST charge is completely determined by the underlying Koszul-Tate differential, and conversely each Koszul-Tate differential gives rise to a classical BRST charge.*

Proof. Let δ be the Koszul-Tate differential associated to some classical BRST charge. Our strategy will be as follows. First, starting from some δ we will construct a classical BRST charge that have the operator δ as its Koszul-Tate differential. The construction is a rather straightforward application of the homological perturbation theory [3]. Then we will show that any two classical BRST charges with the same Koszul-Tate differential are related to each other by a chain of canonical transformations. Either step will crucially exploit the acyclicity of δ in positive resolution degree and pure ghost number.

As in the proof of Theorem 4.5 we split the fields into two classes: the “foreground” and the “background” fields. The background fields are, by definition, the fields with zero pure ghost number and the pure ghost number of foreground fields is strictly positive. Any local functional is given by the integral of a power series in foreground fields and their derivatives. This allows us to split the space of local functionals in the direct sum of two subspaces:

$$\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}'' . \quad (52)$$

The functionals from \mathcal{F}' are at most linear in foreground fields and the functionals from \mathcal{F}'' are at least quadratic in foreground fields. As is shown in Appendix A, the general expression for the Koszul-Tate differential involves only the background fields. Therefore,

$$\delta \mathcal{F}' = \mathcal{F}' , \quad \delta \mathcal{F}'' = \mathcal{F}'' . \quad (53)$$

The subspace \mathcal{F}'' is also closed with respect to the Poisson bracket, i.e.,

$$\{\mathcal{F}'', \mathcal{F}''\} \subset \mathcal{F}'' . \quad (54)$$

Let δ be a Koszul-Tate operator associated to some classical BRST charge. We are looking for a classical BRST charge in the form

$$\Omega_1 = \Omega'_1 + \Omega''_1 , \quad (55)$$

where

$$\mathcal{F}' \ni \Omega'_1 = \int_X v \left(\sum_{s=1}^m \varphi^{i_s} \delta \bar{\varphi}_{i_s} + \sum_{s=1}^{n+1} \bar{\varphi}^{i_s} \delta \varphi_{i_s} \right) \quad \text{and} \quad \Omega''_1 \in \mathcal{F}'' . \quad (56)$$

It is easy to see that

$$\{\Omega'_1, \Omega'_1\} \in \mathcal{F}'' , \quad \{\Omega'_1, \cdot\} = \delta + \dots . \quad (57)$$

Here the dots refer to the terms with positive pure ghost number. It follows from the second relations in (53) and (57) that $\{\Omega'_1, \mathcal{F}''\} \subset \mathcal{F}''$. Notice that the Ω'_1 -part of the classical BRST charge (55) does not contribute to the Koszul-Tate differential as the pure ghost number of the Hamiltonian vector field $\{\Omega'_1, \cdot\}$ is strictly positive. So, if exists, Ω_1 gives rise to the original Koszul-Tate differential δ . The existence is easily proved by applying the machinery of homological perturbation theory. First, we expand Ω_1 according to the resolution degree,

$$\Omega_1 = \sum_{p=0}^{\infty} \Omega_1^{(p)} = \sum_{p=0}^{\infty} (\Omega_1' + \Omega_1''). \quad (58)$$

Note that $\deg \Omega_1' \leq \max(n+1, m+1)$ and $\deg \Omega_1'' = \text{pgh } \Omega_1'' \geq 2$. The master equation $\{\Omega_1, \Omega_1\} = 0$ is then equivalent to the infinite sequence of equations

$$\delta \Omega_1'' = \bar{D} (\Omega_1', \dots, \Omega_1'), \quad (59)$$

where

$$\mathcal{F}'' \ni \bar{D} = \frac{1}{2} \sum_{s=1}^q \sum_{q=0}^p \{ \Omega_1^{(p+s-q)}, \Omega_1 \}^{(s)}, \quad \deg \{ \cdot, \cdot \}^{(s)} = -s. \quad (60)$$

Eqs. (59) are solvable if and only if $\delta \bar{D} = 0$. This follows from the acyclicity of δ in positive resolution degree and pure ghost number (Theorem 4.6). To prove the δ -closedness of \bar{D} , we can proceed by induction on p . For $p = 0$, the functional \bar{D} vanishes and we have the trivial solution $\Omega_1'' = 0$. Suppose that we have satisfied the first p equations of sequence (59) and let

$$R_p = \sum_{s=0}^p \Omega_1^{(s)}$$

be the sum of already known functionals $\Omega_1^{(s)}$. By induction hypothesis

$$\{R_p, R_p\} = 2 \bar{D} + \dots,$$

where the unwritten terms are of resolution degree $> p$. Extracting the leading term of the Jacobi identity $\{R_p, \{R_p, R_p\}\} = 0$, we find that $\delta \bar{D} = 0$. Thus, Eq. (59) possesses a solution and, by virtue of (53), we can choose Ω_1'' to be an element of \mathcal{F}'' .

From the consideration above it follows that the first two terms $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$ of the classical BRST charge (58) are unambiguously determined by the Koszul-Tate differential δ . By contrast, the higher order terms $\Omega_1^{(p)}$, $p \geq 2$, are not so unique as Eq. (59) defines them only up to a δ -coboundary. Let us show that this ambiguity can be entirely absorbed by an appropriate canonical transformation of \mathcal{F} . Consider two different classical BRST charges Ω_1 and $\bar{\Omega}_1$ that share one and the same Koszul-Tate differential δ and coincide to each other up to the k -th

order in resolution degree. In view of the comment above, this means $k \geq 1$. By (59) the difference $\overline{\Omega}_1^{(k+1)} - \Omega_1^{(k+1)}$ is δ -closed. Since both the resolution degree and pure ghost number of the difference are positive, we can write

$$\overline{\Omega}_1^{(k+1)} = \Omega_1^{(k+1)} + \delta\Phi$$

for some $\Phi \in \mathcal{F}''$ of resolution degree $k+2$. The functional Φ generates a well-defined canonical transformation $e^{\{\Phi, \cdot\}} : \mathcal{F} \rightarrow \mathcal{F}$. (Since $\Phi \in \mathcal{F}''$, only finitely many terms of the exponential series $e^{\{H, \cdot\}} = 1 + \{H, \cdot\} + \dots$ contribute to each order in pure ghost number.) Applying this transformation to $\overline{\Omega}_1$, we get a new solution to the master equation

$$\tilde{\Omega}_1 = e^{\{\Phi, \cdot\}} \overline{\Omega}_1 = \overline{\Omega}_1 - \delta\Phi + \dots$$

By definition, $\tilde{\Omega}_1$ coincides with Ω_1 at least to order $k+1$ and gives rise to the same Koszul-Tate operator. So, the classical BRST charges Ω_1 and $\overline{\Omega}_1$ appear to be canonically equivalent to the $(k+1)$ -st order. The same arguments, being applied to the pair $\Omega_1, \tilde{\Omega}_1$, show that Ω_1 is canonically equivalent to $\tilde{\Omega}_1$, and hence to $\overline{\Omega}_1$, up to order $k+2$. Repeating these arguments once and again, we infer the canonical equivalence of Ω_1 and $\overline{\Omega}_1$ at all orders in resolution degree. □

Though useful (e.g. for defining the conservation laws and rigid symmetries), the classical BRST charge carries no valuable information about quantum properties of the system. The quantum properties are determined by the higher order terms in the expansion of the “total” BRST charge (10) with respect to the momentum degree. Letting all the higher terms to be zero, which is admissible, and applying the general quantization procedure [5] to the BRST charge $\Omega = \Omega_1$, one can see that the probability amplitude on the space of fields is given by the singular distribution supported on the solutions to the classical equations of motion. As a result, the quantum averages of physical observables are reduced to their classical values with no quantum fluctuations. Thus, to get a nontrivial quantization, one has to admit higher order terms in the BRST charge. These can be considered as a deformation of the classical BRST charge by higher order terms in momentum degree. In the rest of the section, we are going to clarify the structure of such deformations by making use of previously obtained knowledge of the local BRST cohomology. To this end, consider the decomposition of the BRST charge into the sum

$$\Omega = \Omega_1 + \mathbf{\Omega}, \quad \mathbf{\Omega} = \sum_{s=2}^{\infty} \Omega_s, \quad \text{Deg } \Omega_s = s. \quad (61)$$

It is the functional Ω that is regarded as a deformation of the classical BRST charge Ω_1 . Given Ω_1 , the master equation $\{\Omega, \Omega\} = 0$ takes the form of the Maurer-Cartan equation

$$s_0\Omega = \{\Omega, \Omega\} \quad (62)$$

with respect to the classical BRST differential

$$s_0 = \{\Omega_1, \cdot\} = \delta + \dots, \quad s_0^2 = 0. \quad (63)$$

The deformation Ω is said to be of order p if its expansion in homogeneous components starts with a term of momentum degree p ,

$$\Omega = \sum_{m=p}^{\infty} \Omega_m.$$

Two deformations Ω and Ω' are said to be equivalent if they are related to each other by a canonical transformation, i.e., there exists a local functional Φ with $\text{Deg } \Phi \geq 2$ such that

$$e^{\{\Phi, \cdot\}}(\Omega_1 + \Omega) = \Omega_1 + \Omega'. \quad (64)$$

A *trivial deformation* is a deformation that is equivalent to zero. We say that a deformation of order p is *strict* if it is not equivalent to a deformation of order $> p$. Let $\text{Def}(\delta) = \bigcup_{m \geq 2} \text{Def}_m(\delta)$ denote the set of all solution to the Maurer-Cartan equation (62) modulo canonical transformations; here the subset $\text{Def}_m(\delta)$ consists of the strict deformations of order m . The deformations of $\text{Def}_2(\delta)$ are called *regular* and the deformations of order > 2 are called *singular*.

On substituting the expansion (61) into (62), we arrive at the sequence of equations

$$\begin{aligned} s_0\Omega_m &= 0, & m &= p, \dots, 2p-2; \\ s_0\Omega_m &= -\frac{1}{2} \sum_{n=p}^{m-2} \{\Omega_{m-n}, \Omega_{n+1}\}, & m &> 2p-2. \end{aligned} \quad (65)$$

The first equation of the sequence, $s_0\Omega_p = 0$, means that the leading term of the deformation is a s_0 -cocycle. For a strict deformation this cocycle is necessarily nontrivial for if $\Omega = s_0\Phi$, then the canonical transformation (64) gives the BRST charge Ω' with $\text{Deg } \Omega' > p$. We have the map

$$\text{Def}_m(\delta) \rightarrow H_m^1(s_0|d)^\Delta$$

associating to each deformation its leading term. In general this map is neither injective nor surjective. The latter fact implies that not all s_0 -closed functionals of ghost number 1 and momentum degree $p \geq 2$ can serve as leading terms of p -th order deformations. Consider for example a regular deformation, that is, a strict deformation of order 2. It follows from (65)

that at the third order in momentum degree the leading term of such a deformation, Ω_2 , obeys the equation

$$s_0\Omega_3 = \frac{1}{2}\{\Omega_2, \Omega_2\}. \quad (66)$$

By the Jacobi identity the right hand side of this equation is s_0 -closed whenever Ω_2 is a s_0 -cocycle. But the equation requires the Poisson square of Ω_2 to be s_0 -exact. In other words, the Maurer-Cartan equation possesses a solution at the third order in momentum degree iff

$$\{[\Omega_2], [\Omega_2]\} = [\{\Omega_2, \Omega_2\}] = 0 \in H_3^2(s_0|d)^\Delta.$$

At the next step we get the equation

$$D\Omega_4 = \{\Omega_3, \Omega_2\}.$$

Again, the right hand side of this equation is s_0 -closed but not generally s_0 -exact. The class $\{[\Omega_3, \Omega_2]\}$, where Ω_3 is defined by Eq. (66), is known as the Massey cube of the class $[\Omega_2]$. It is usually denoted by $\langle[\Omega_2], [\Omega_2], [\Omega_2]\rangle$. Vanishing of the class $\langle[\Omega_2], [\Omega_2], [\Omega_2]\rangle$ is thus the necessary and sufficient condition for the Maurer-Cartan equation (62) to admit a solution at the fourth order in momentum degree. It should be emphasized that the Massey cube can only be defined for a class $[\Omega_2]$ with vanishing “Massey square” $\langle[\Omega_2], [\Omega_2]\rangle = \{[\Omega_2], [\Omega_2]\}$ and Eq. (66) defines Ω_3 up to an arbitrary s_0 -cocycle. So, one can regard the assignment $[\Omega_2] \mapsto \langle[\Omega_2], [\Omega_2], [\Omega_2]\rangle$ as a partially defined and multivalued map from $H_3^2(s_0|d)^\Delta$ to $H_4^2(s_0|d)^\Delta$. Proceeding in the same manner one can see that the class $[\Omega_2] \in H_2^1(s_0|d)^\Delta$ extends to an element of $\text{Def}_2(\delta)$ iff all the Massey powers of $[\Omega_2]$ can be made zero,

$$\underbrace{\langle[\Omega_2], [\Omega_2], \dots, [\Omega_2]\rangle}_n \ni 0 \quad \forall n = 2, 3, \dots \quad (67)$$

For a general definition of the Massey products in the category of differential graded Lie algebras we refer the reader to [26], [27].

A similar analysis applies to singular deformations. In particular, the first equation in (65) shows that the square $\{[\Omega_p], [\Omega_p]\} \in H_{2p-1}^2(s_0|d)^\Delta$ of the leading term of a p -th order deformation has to vanish. This ensures the existence of a solution to (62) at order $2p - 1$. The study of higher order obstructions to solvability of the Maurer-Cartan equation appears to be much more involved than in the case of regular deformations, but it can still be formulated in terms of Massey-like products [27]. We will not go into details here.

Turning back to the regular deformations, we see that the existence of a solution to the Maurer-Cartan equation (62) is generally obstructed by elements of the cohomology groups $H_m^2(s_0|d)^\Delta$, $m > 2$. By Theorem 4.8 all these groups are isomorphic to the corresponding

groups $H_m^2(\delta|d)_0^\Delta$. Furthermore, the isomorphism (47) can be regarded as an isomorphism of graded Lie algebras if we endow $H_m^2(\delta|d)_0^\Delta$ with the pulled-back Lie bracket given by the r.h.s. of (48)⁸. This allows us to rewrite the condition (67) in the following form:

$$H_{n+1}^2(\delta|d)_0^\Delta \supset \underbrace{\langle \Lambda, \Lambda, \dots, \Lambda \rangle}_n \ni 0 \quad \forall n = 2, 3, \dots, \quad (68)$$

$\Lambda = \pi([\Omega_2])$ being the Lagrange structure underlying the BRST charge Ω . Thus, we conclude that a Lagrange structure Λ gives rise to a regular deformation of a classical BRST charge iff all its Massey powers can be made zero. In particular, vanishing of the Massey square of Λ amounts to its integrability by Definition (5.1).

Actually, only a finite number of classes of the sequence (68) may present true obstacles to the existence of Ω , as for all $n > \Delta + 1$, $H_{n+1}^2(\delta|d)_0^\Delta = 0$ by Theorem 4.4. For example, if $\Delta = 1$ (the case of mechanical systems) each integrable Lagrange structure gives rise to a regular deformation, and conversely each nontrivial deformation is necessarily regular and is defined by some integrable Lagrange structure. In our next paper we will show that there is a one-to-one correspondence between the integrable Lagrange structures in one dimension and the weak Poisson structures introduced in [4]. This correspondence allows one to relate the path-integral quantization of not necessarily Lagrangian mechanical systems with their deformation quantization. For actual field theories ($\Delta > 0$) singular deformations may exist as well. The above analysis shows, for instance, that any element of $H_p^1(\delta|d)_0^\Delta$ with $p > (\Delta + 3)/2$ generates a singular deformation of order p . The quantization by means of singular deformations, having no analogue in Lagrangian field theory, looks rather intriguing, to say the least. Its interpretation in terms of deformation quantization remains obscure to us, though.

We summarize the main results of this section by the following

Theorem 7.2. *Given a Koszul-Tate differential δ and a Lagrange structure $\Lambda \in H_2^1(\delta|d)_0^\Delta$ obeying the finite set of conditions*

$$\underbrace{\langle \Lambda, \Lambda, \dots, \Lambda \rangle}_n \ni 0 \quad \forall n = 2, 3, \dots, \Delta + 1,$$

there exists a local BRST charge $\Omega = \sum_{k=1}^\infty \Omega_k$ such that

$$\{\Omega_1, \cdot\} = \delta + (\text{terms of resolution degree} \geq 0) \quad \text{and} \quad \pi([\Omega_2]) = \Lambda.$$

In general, the Ω is not unique even when considered up to a canonical equivalence.

⁸It should be noted that the pulled-back Lie bracket does not coincide with the naive Poisson bracket on $H_m^2(\delta|d)_0^\Delta$ unless the resolution degrees of multiplied classes are less than or equal to 1.

8. CONCLUSION

In this paper, we have extended the main theorems on the local BRST cohomology beyond the scope of Lagrangian dynamics. Let us comment on the most curious peculiarities of the non-Lagrangian BRST complex and discuss further perspectives.

In the first place, we see that the groups $H_1^0(\delta|d)_0^\Delta$ and $H_0^{-1}(\delta|d)_0^\Delta$, whose elements are identified with nontrivial rigid symmetries and characteristics, are not generally isomorphic to each other unlike it happens in Lagrangian dynamics. These groups, however, combine within the unifying group $H^{(1)}(\delta|d)_0^\Delta = \bigoplus_g H_{g+1}^g(\delta|d)_0^\Delta$. The group $H^{(1)}(\delta|d)_0^\Delta$ is shown to have the structure of a graded Lie algebra of depth -1. It then follows that the homogeneous subgroup $H_1^0(\delta|d)_0^\Delta$ of ghost number zero is a subalgebra in $H^{(1)}(\delta|d)_0^\Delta$ (isomorphic to the Lie algebra of rigid symmetries) and the bottom subspace $H_0^{-1}(\delta|d)_0^\Delta$ is a module over the Lie algebra $H_1^0(\delta|d)_0^\Delta$ with respect to the adjoint action (the symmetries act on the characteristics). The next homogeneous subspace of the unifying group is $H_2^1(\delta|d)_0^\Delta$. It is understood as the space of Lagrange structures. Again, by a purely algebraic reason, the Lie bracket of a Lagrange structure and a characteristic is a rigid symmetry. Thus, each Lagrange structure defines a map from the space of characteristics to the space of rigid symmetries. In the Lagrangian case, the space $H_2^1(\delta|d)_0^\Delta$ contains a distinguished element - the canonical Lagrange structure - for which the aforementioned map is injective and covers all the symmetries of the corresponding action functional. This is the content of the first Noether's theorem. For an arbitrary Lagrange structure, the map is neither injective nor surjective, so that the symmetries and conservation laws cannot be so tightly related to each other beyond the class of Lagrangian field theories.

Besides being a module over the Lie algebra of rigid symmetries, the space of conservation laws of any Lagrangian field theory carries its own Lie algebra structure with respect to the Dickey bracket. Like the Noether isomorphism, the Dickey bracket owes its existence to the canonical Lagrange structure. When trying to extend the Dickey bracket beyond the Lagrangian setting, one naturally arrives at the concept of integrability of Lagrange structure. Namely, a Lagrange structure Λ is said to be integrable if it satisfies the Maurer-Cartan equation $[\Lambda, \Lambda] = 0$. Given an integrable Lagrange structure, one can define a non-Lagrangian counterpart of the Dickey bracket as the derived bracket of characteristics (50) and the same Lagrange structure defines the Lie algebra homomorphism (49). The notion of a Lagrange structure appears thus to be a principal connecting-link for the groups of symmetries and conservation laws. The interpretation of the higher homogeneous subgroups of $H^{(1)}(\delta|d)_0^\Delta$ remains unclear at present, maybe excluding the group $H_3^2(\delta|d)_0^\Delta$ that can be regarded as the obstruction space

for integrability of Lagrange structures. Further studies of the local BRST cohomology will hopefully shed more light on the utility of these groups.

The study of symmetries and conservation laws of partial differential equations is a broad area of mathematical physics with numerous applications and well developed methodology. It seems that similar methods apply to computation of the group $H_2^1(\delta|d)_0^\Delta$ of Lagrange structures, though it can be a difficult problem, in general. In any case, every single Lagrange structure gives a valuable bit of information about classical dynamics and, what is important, defines a reasonable quantum theory. As far as the quantization problem is concerned, one is usually interested in Lagrange structures subject to one or another set of physical conditions, rather than in knowing the entire space $H_2^1(\delta|d)_0^\Delta$. For example, if some fundamental symmetries of classical dynamics are expected to survive at the quantum level, the same invariance conditions should be imposed on the Lagrange structure. (Notice that the space of Lagrange structures $H_2^1(\delta|d)_0^\Delta$ is a module over the Lie algebra of rigid symmetries $H_1^0(\delta|d)_0^\Delta$). These conditions together with some other physical requirements may strongly restrict the possible choice of a Lagrange structure or even make it unique. An additional set of conditions comes from requiring the existence of a local BRST charge. Unlike the usual BV formalism for Lagrangian gauge theories, the locality of equations of motion and a compatible Lagrange structure Λ does not ensure the existence of a local BRST charge. In order for such a charge to exist, it is necessary and sufficient that all the Massey powers $\langle \Lambda, \Lambda, \dots, \Lambda \rangle$ of the cohomology class $\Lambda \in H_2^1(\delta|d)_0^\Delta$ to vanish. In particular, the vanishing of the Massey square reproduces the integrability condition.

Finally, there can exist higher-order deformations of the classical BRST charge that are governed by elements of the groups $H_m^1(\delta|d)_0^\Delta$ with $m > 2$. These deformations, called singular, are unrelated to any Lagrange structure and their relevance to the path-integral quantization of non-Lagrangian field theories invites further studies.

Acknowledgments. We dedicate this paper to the 70th birthday of Igor Victorovich Tyutin, who contributed so much to theory of gauge systems, and who so much helped many of his colleagues, in many ways, during many years.

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APPENDIX A. THE PROOFS OF TWO AUXILIARY STATEMENTS

Any Hamiltonian vector field $X_H = \{H, \cdot\}$ generated by a local functional $H \in \mathcal{F}$ is decomposed into the sum

$$X_H = V_H + U_H$$

of two variational vector fields⁹

$$\begin{aligned} V_H &= \sum_s \int_X v \left((-1)^{s(\varepsilon(H)-1)-1} \frac{\delta H}{\delta \bar{\varphi}^{i_s}} \frac{\delta}{\delta \varphi_{i_s}} + (-1)^{(s-1)\varepsilon(H)} \frac{\delta H}{\delta \varphi^{i_{s-1}}} \frac{\delta}{\delta \bar{\varphi}_{i_{s-1}}} \right), \\ U_H &= \sum_s \int_X v \left((-1)^{s\varepsilon(H)} \frac{\delta H}{\delta \varphi_{i_s}} \frac{\delta}{\delta \bar{\varphi}^{i_s}} + (-1)^{s(\varepsilon(H)-1)+\varepsilon(H)} \frac{\delta H}{\delta \bar{\varphi}_{i_{s-1}}} \frac{\delta}{\delta \varphi^{i_{s-1}}} \right) \end{aligned} \quad (69)$$

such that

$$\text{pgh } V_H \geq 0, \quad \deg U_H \geq 0. \quad (70)$$

To prove Proposition 2.1 consider the Hamiltonian vector field $X_\Omega = V_\Omega + U_\Omega$ generated by the BRST charge Ω . As the resolution degree of U_Ω is bounded from below by zero, it remains to evaluate the low bound for V_Ω . Applying the identity (3) yields

$$\deg V_\Omega = \text{pgh } V_\Omega + \text{Deg } V_\Omega - \text{gh } V_\Omega. \quad (71)$$

On the other hand,

$$\text{gh } V_\Omega = 1, \quad \text{Deg } V_\Omega = \text{Deg } \Omega - 1 \geq 0. \quad (72)$$

Combining (70), (71), and (72) we get

$$\deg V_\Omega = \text{pgh } V_\Omega + \text{Deg } \Omega - 2 \geq -1. \quad (73)$$

The last inequality shows two things: (i) only the classical BRST charge Ω_1 contributes to δ and (ii) $\text{pgh } \delta = 0$. From the latter property and the definition of V_Ω it also follows that the Koszul-Tate differential δ is completely determined in terms of fields of pure ghost number zero (background fields).

Similar arguments allows one to prove the equality (48). Let a and b be relative δ -cocycles representing the classes $\alpha, \beta \in H^{(1)}(\delta|d)_0^\Delta$. By the definition of the isomorphism (47), the classes $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$ are represented by relative s_0 -cocycles of the form $a + A$ and $b + B$ with $\text{pgh } A > 0$ and $\text{pgh } B > 0$. To prove the equality in question we only need to show that the resolution-degree-one part of the Poisson bracket

$$\{a + A, b + B\} = \{a, b\} + \{a, B\} + \{A, b + B\} \quad (74)$$

⁹All the variational derivatives act on the left.

is given by $\{a, b\}$. The equality $\deg \{a, b\} = 1$ is obvious as a and b belong to the Lie algebra \mathcal{Z} , see Sec. 5. What is left is to show that the resolution degree of the other two terms in (74) is greater than 1. Applying (69) we can write $\{a, B\} = V_a B + U_a B$, where

$$\text{pgh } V_a B \geq \text{pgh } B > 0, \quad \deg U_a B \geq \deg B > 0. \quad (75)$$

On the other hand, for any $[c] \in H_{g+1}^g(s_0|d)^\Delta$ we have

$$\deg c = \text{pgh } c + 1.$$

In particular, this equality is true for (each term of) the Poisson bracket (74). Combining the last fact with the second inequality (75) we get

$$\text{pgh } U_a B \geq \text{pgh } B > 0.$$

Together with the first inequality (75) this yields $\text{pgh } \{a, B\} \geq \text{pgh } B > 0$. In the same way one can verify that $\text{pgh } \{A, b + B\} \geq \text{pgh } A > 0$.

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